[三角関数の有理式]4H1 後半

1. 次の不定積分を求めよ。

(1) $I=\int \frac{1}{2\cos x+3\sin x+2}dx$

(2) $I=\int \frac{1}{\cos x+2\sin x+3}dx$

[定積分:偶関数と奇関数]

2. 次の定積分の値を求めよ。

(1) $I=\int_{-2}^{2}\left(x^5-2x^3+3x^2+5x+1\right)dx$

(2) $I=\int_{-1}^{1}\left(\frac{x\cos x+1}{x^2+1}\right)dx$

[定積分:基本]

3. 次の定積分の値を求めよ。

(1) $\int_{-2}^{1}x^2\,dx$

(2) $\int_{0}^{7}\frac{1}{\sqrt[3]{x+1}}dx$

(3) $\int_{0}^{1}\frac{1}{x+1}dx$

(4) $\int_{2}^{3}e^{2x-4}\,dx$

(5) $\int_{0}^{\frac{\pi}{4}}\sin 3x\,dx$

(6) $\int_{0}^{\frac{\pi}{4}}\frac{1}{\cos^2 x}dx$

(7) $\int_{0}^{3}\frac{1}{x^2+3}dx$

(8) $\int_{0}^{\sqrt{2}}\frac{1}{\sqrt{4-x^2}}dx$

[三角関数の有理式]4H1 後半

1. 次の不定積分を求めよ。

(1) $I=\int \frac{1}{2\cos x+3\sin x+2}dx \quad u=\tan\frac{x}{2}$ とおくと、

$\cos x=\frac{1-u^2}{1+u^2},\ \sin x=\frac{2u}{1+u^2},\ dx=\frac{2}{1+u^2}du$ より

$$I=\int \frac{1}{\left(2\cdot\frac{1-u^2}{1+u^2}+3\cdot\frac{2u}{1+u^2}+2\right)}\frac{2}{1+u^2}du$$

$$=\int \frac{2}{\underbrace{2\cdot(1-u^2)+3\cdot 2u+2(1+u^2)}_{4+6u=2(3u+2)}}du$$

$$=\int \frac{1}{3u+2}du=\frac{1}{3}\int \frac{3}{3u+2}du=\frac{1}{3}\log|3u+2|+C$$

$$=\frac{1}{3}\log\left|3\tan\frac{x}{2}+2\right|+C$$

(2) $I=\int \frac{1}{\cos x+2\sin x+3}dx \quad u=\tan\frac{x}{2}$ とおくと、

$\cos x=\frac{1-u^2}{1+u^2},\ \sin x=\frac{2u}{1+u^2},\ dx=\frac{2}{1+u^2}du$ より

$$I=\int \frac{1}{\left(\frac{1-u^2}{1+u^2}+2\cdot\frac{2u}{1+u^2}+3\right)}\frac{2}{1+u^2}du$$

$$=\int \frac{2}{\underbrace{(1-u^2)+2\cdot 2u+3\cdot(1+u^2)}_{2u^2+4u+4=2(u^2+2u+2)=2\{(u+1)^2+1\}}}du$$

$$=\int \frac{1}{(u+1)^2+1}du=\mathrm{Tan}^{-1}(u+1)+C$$

$$=\mathrm{Tan}^{-1}\left(\tan\frac{x}{2}+1\right)+C$$

[定積分:偶関数と奇関数]

2. 次の定積分の値を求めよ。

(1) $I=\int_{-2}^{2}(x^5-2x^3+3x^2+5x+1)dx$

x^5-2x^3+5x は奇関数より

$$\int_{-2}^{2}(x^5-2x^3+3x^2+5x)dx=0$$

また、$3x^2+1$ は偶関数だから

$$I=2\int_0^2(3x^2+1)dx=2\left[x^3+x\right]_0^2=2(8+2)=20$$

(2) $I=\int_{-1}^{1}\left(\frac{x\cos x+1}{x^2+1}\right)dx=\int_{-1}^{1}\frac{x\cos x}{x^2+1}dx+\int_{-1}^{1}\frac{1}{x^2+1}dx$

$\frac{x\cos x}{x^2+1}$ は奇関数だから $\int_{-1}^{1}\frac{x\cos x}{x^2+1}dx=0$ より

$$I=\int_{-1}^{1}\underbrace{\frac{1}{x^2+1}}_{偶関数}dx=2\int_0^1\frac{1}{x^2+1}dx=2\left[\mathrm{Tan}^{-1}x\right]_0^1$$

$$=2(\mathrm{Tan}^{-1}1-0)=2\left(\frac{\pi}{4}-0\right)=\frac{\pi}{2}$$

[定積分:基本]

3. 次の定積分の値を求めよ。

(1) $\int_{-2}^{1}x^2dx=\frac{1}{3}\left[x^3\right]_{-2}^{1}=\frac{1}{3}\left\{1^3-(-2)^3\right\}=\frac{9}{3}=3$

(2) $\int_0^7\frac{1}{\sqrt[3]{x+1}}dx=\int_0^7(x+1)^{\frac{-1}{3}}dx=\frac{3}{2}\left[(x+1)^{\frac{2}{3}}\right]_0^7$

$$=\frac{3}{2}\left(8^{\frac{2}{3}}-1^{\frac{2}{3}}\right)=\frac{3}{2}\left\{(2^3)^{\frac{2}{3}}-1^{\frac{2}{3}}\right\}=\frac{3}{2}(2^2-1)=\frac{9}{2}$$

(3) $\int_0^1\frac{1}{x+1}dx=\left[\log|x+1|\right]_0^1=\left(\log 2-\underbrace{\log 1}_{0}\right)=\log 2$

(4) $\int_2^3 e^{2x-4}dx=\frac{1}{2}\left[e^{2x-4}\right]_2^3=\frac{1}{2}\left(e^2-\underbrace{e^0}_{1}\right)=\frac{1}{2}(e^2-1)$

(5) $\int_0^{\frac{\pi}{4}}\sin 3x\,dx=\frac{-1}{3}\left[\cos 3x\right]_0^{\frac{\pi}{4}}=\frac{-1}{3}\left(\cos\frac{3\pi}{4}-\underbrace{\cos 0}_{1}\right)$

$$=\frac{-1}{3}\left(\frac{-\sqrt{2}}{2}-1\right)=\frac{1}{3}\cdot\frac{\sqrt{2}+2}{2}=\frac{\sqrt{2}+2}{6}$$

(6) $\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2 x}dx=\left[\tan x\right]_0^{\frac{\pi}{4}}=\underbrace{\tan\frac{\pi}{4}}_{1}-\underbrace{\tan 0}_{0}=1$

(7) $\int_0^3\frac{1}{x^2+3}dx=\frac{1}{\sqrt{3}}\left[\mathrm{Tan}^{-1}\frac{x}{\sqrt{3}}\right]_0^3$

$$=\frac{1}{\sqrt{3}}\left(\mathrm{Tan}^{-1}\frac{3}{\sqrt{3}}-\underbrace{\mathrm{Tan}^{-1}0}_{0}\right)$$

$$=\frac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\sqrt{3}=\frac{1}{\sqrt{3}}\cdot\frac{\pi}{3}=\frac{\sqrt{3}\pi}{9}$$

(8) $\int_0^{\sqrt{2}}\frac{1}{\sqrt{4-x^2}}dx=\left[\mathrm{Sin}^{-1}\frac{x}{2}\right]_0^{\sqrt{2}}$

$$=\mathrm{Sin}^{-1}\frac{\sqrt{2}}{2}-\underbrace{\mathrm{Sin}^{-1}0}_{0}=\frac{\pi}{4}$$